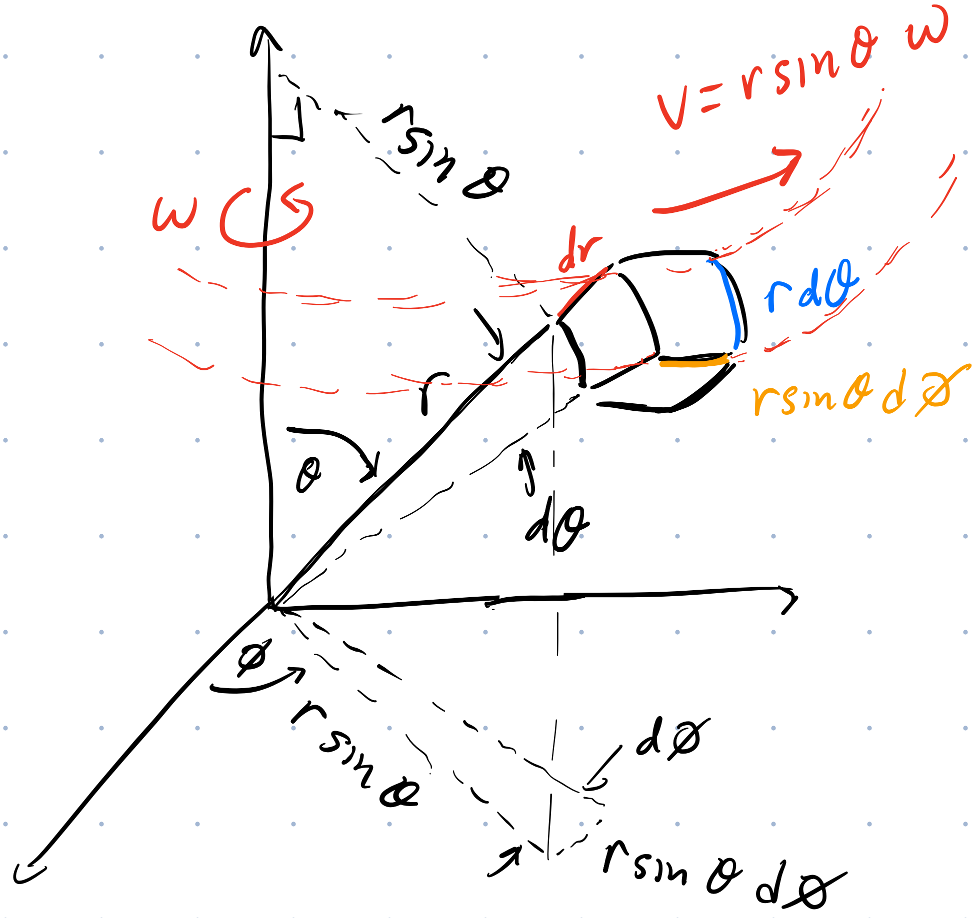
PHYS 301 Assign. #4 Sol'ns Nov. 11, 2024

1. (a)



In spherical coords volume element

$$d\tau = r^2 \sin\theta \, d\theta \, d\phi \, dr$$

Charge contained in volume element is

$$dQ = \rho \, d\tau = \rho r^2 \sin\theta \, d\theta \, d\phi \, dr$$

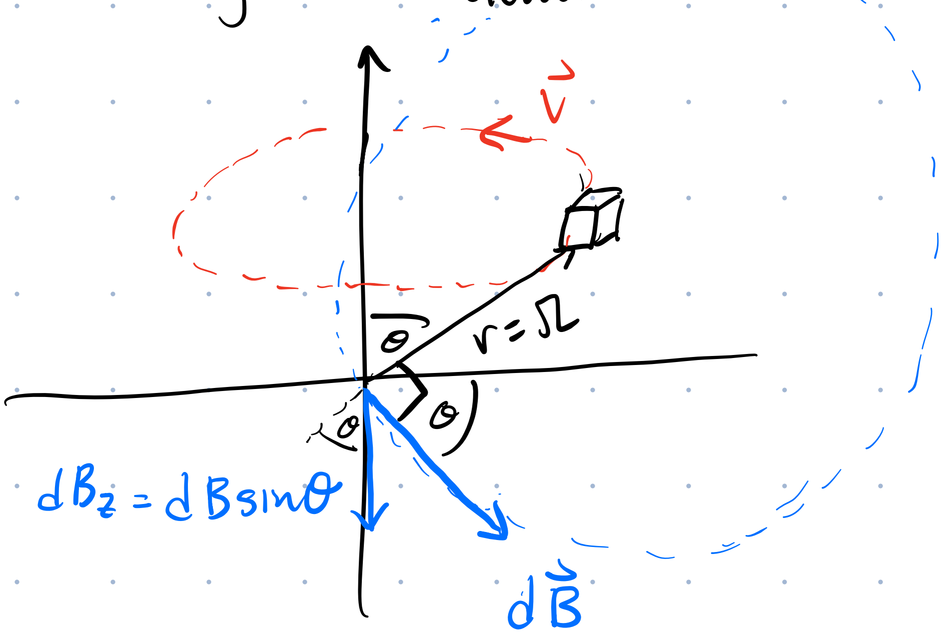
$$I\,d\ell = \frac{dQ}{dt}\,d\ell = dQ\,\frac{d\ell}{dt} = dQ\,v$$

$$\therefore I\,d\ell = \left(\rho r^2 \sin\theta\, d\theta\, d\phi\, dr\right)\left(r\omega\sin\theta\right)$$

$$\boxed{\therefore I\,d\ell = \rho\omega r^3 \sin^2\theta\, d\theta\, d\phi\, dr}$$

(b) Biot-Savart Law: $dB = \frac{\mu_0}{4\pi}\,\frac{I\,d\vec{\ell}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$

For the rotating volume element $d\tau$



By symmetry, only the z-component of dB will survive the ϕ integration.

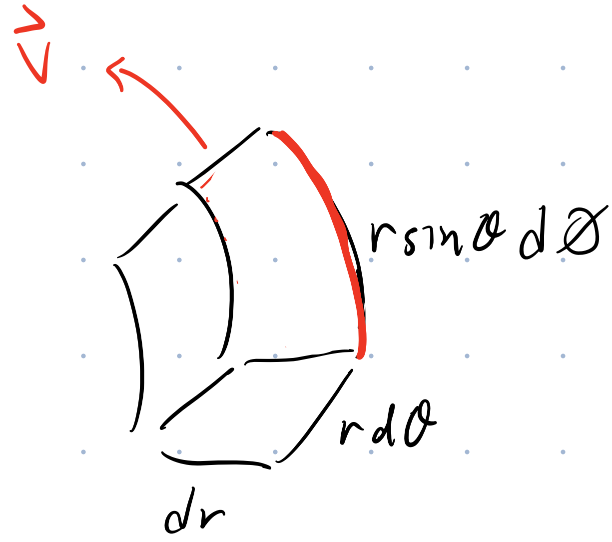
- Furthermore, b/c our "field pt" is the origin, $\mathscr{r} = r$.
- Note: $\hat{\mathscr{r}} \perp d\vec{l}$ $\therefore$ $|d\vec{l} \times \hat{\mathscr{r}}| = dl$

$$\therefore dB_z \equiv dB = \frac{\mu_0}{4\pi} \frac{\rho \omega r^3 \sin^3\theta \, d\theta \, d\phi \, dr}{r^2}$$

drop the z subscript

$$\boxed{\therefore dB = \frac{\mu_0}{4\pi} \rho \omega r \sin^3\theta \, d\theta \, d\phi \, dr}$$

(c)



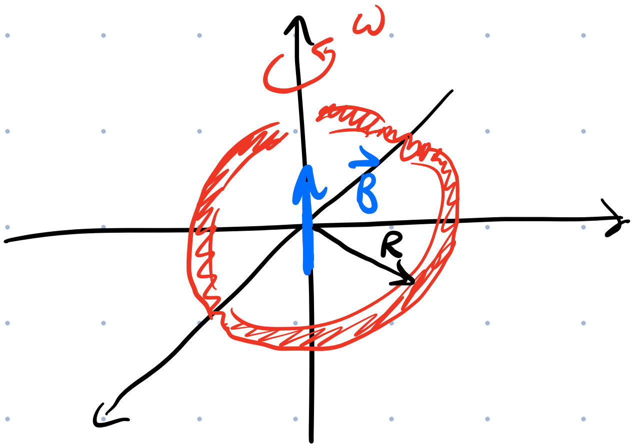
$$Q = \rho\, d\tau = \rho\, r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$\lambda = \frac{Q}{r\sin\theta\, d\phi} = \boxed{\rho\, r\, d\theta\, dr}$$

$$\therefore dB = \frac{\mu_0}{4\pi}\omega \lambda \sin^3\theta\, d\phi$$

Setting $\theta = \frac{\pi}{2}$ places $d\tau$ in xy plane.
The ϕ integral gives $\vec{B}$ due to a spinning ring of charge.

$$B = \int_0^{2\pi} \frac{\mu_0}{4\pi}\omega\lambda\, d\phi = \boxed{\frac{\mu_0 \omega \lambda}{2} = B_{ring}}$$



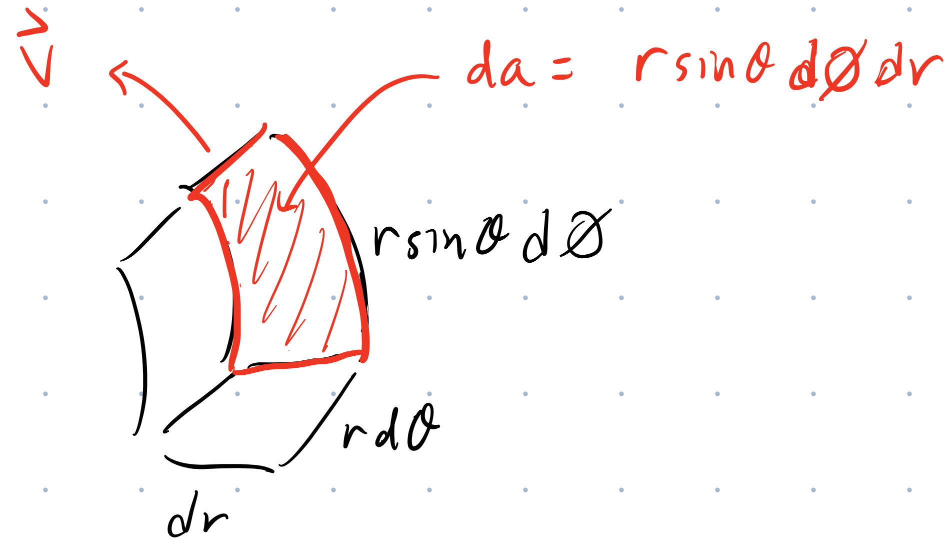
Note: if ring radius is R, then:

$$B_{ring} = \frac{\mu_0 \omega \lambda}{2} \frac{R}{R} = \frac{\mu_0 \overbrace{R\omega}^{v} \lambda}{2R}$$

$$= \frac{\mu_0 \overbrace{v\lambda}^{I}}{2R}$$

$$\boxed{\therefore B_{ring} = \frac{\mu_0 \omega \lambda}{2} = \frac{\mu_0 I}{2R}}$$

(d)



$$dQ = \rho\, d\tau = \rho r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$\sigma_{disk} = \frac{dQ}{da} = \frac{dQ}{r\sin\theta\, d\phi\, dr} = \boxed{\rho r\, d\theta}$$

$$\boxed{\therefore\ dB = \frac{\mu_0}{4\pi} \omega \sigma_{disk} \sin^3\theta\, d\phi\, dr}$$

Set $\theta = \pi/2$ & integrate to find B_{disk}.

$$B_{disk} = \int_{r=0}^{R} \int_{\phi=0}^{2\pi} \frac{\mu_0}{4\pi} \omega \sigma_{disk}\, d\phi\, dr$$

$$\boxed{\therefore\ B_{disk} = \frac{\mu_0}{2}\,\omega\,\sigma_{disk}\,R}$$

(e) $$B_{sphere} = \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\mu_0}{4\pi}\,\rho\,\omega\,r\sin^3\theta\,dr\,d\theta\,d\phi$$

$$= \frac{\mu_0\rho\omega}{2}\underbrace{\int_{r=0}^{R} r\,dr}_{\frac{R^2}{2}}\ \underbrace{\int_{\theta=0}^{\pi}\sin^3\theta\,d\theta}_{4/3\ \text{(see below)}}$$

$$\int_0^{\pi}\sin^3\theta\,d\theta = \int_0^{\pi}\underbrace{\sin^2\theta}_{1-\cos^2\theta}\ \underbrace{\sin\theta\,d\theta}_{-d(\cos\theta)}$$

$$= -\int_0^{\pi}(1-\cos^2\theta)\,d(\cos\theta)$$

sub $u=\cos\theta$ $\quad$ $\theta=0,\ u=1$

$\theta=\pi,\ u=-1$

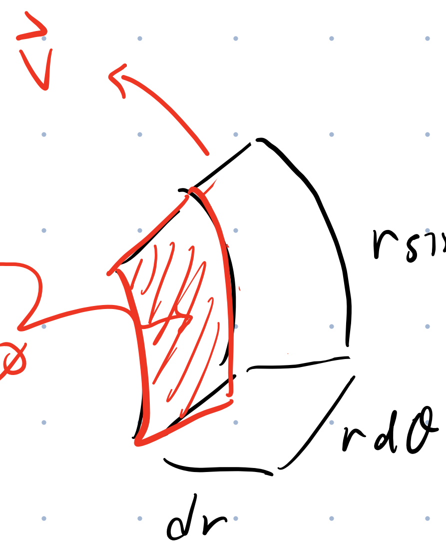
$$= -\int_{1}^{-1} (1-u^2)\,du = \int_{-1}^{1} (1-u^2)\,du$$

$$= \left(u - \frac{u^3}{3}\right)\Bigg|_{-1}^{1} = 2 - \frac{2}{3} = \frac{6-2}{3} = \frac{4}{3}$$

$$\therefore B_{sphere} = \frac{\mu_0 \rho \omega}{\cancel{2}} \frac{R^2}{\cancel{2}} \frac{\cancel{4}}{3}$$

$$\boxed{\therefore B_{sphere} = \frac{1}{3}\mu_0 \rho \omega R^2}$$

(f)



$da = r^2 \sin\theta\, d\theta\, d\phi$

$$dQ = \rho\, d\tau$$

$$\sigma_{shell} = \frac{dQ}{da} = \frac{\rho\, r^2 \sin\theta\, d\theta\, d\phi\, dr}{r^2 \sin\theta\, d\theta\, d\phi} = \boxed{\rho\, dr}$$

$$\therefore\ dB_{shell} = \frac{\mu_0}{4\pi}\omega\, \sigma_{shell}\, r \sin^3\theta\, d\theta\, d\phi$$

$$\therefore\ B_{shell} = \frac{\mu_0}{4\pi}\omega\, \sigma_{shell}\, r \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi} \underbrace{\int_{\theta=0}^{\pi} \sin^3\theta\, d\theta}_{4/3}$$

$$\boxed{\therefore\ B_{shell} = \frac{2}{3}\mu_0 \omega\, \sigma_{shell}\, r}$$

2. Start w/

$$L = I_{rot}\,\omega = \omega \int r^2\, dm$$

write $dm = \rho_m\, d\tau$ s.t. :

$$\boxed{L = \rho_m \omega \int r^2\, d\tau} \quad \text{(i)}$$

Next, work on :

$$\mu = \pi \int r^2\, dI$$

$$dI = \frac{dQ}{T} = \frac{\rho_q\, d\tau}{(2\pi/\omega)} = \frac{\rho_q \omega\, d\tau}{2\pi}$$

$$\therefore \mu = \cancel{\pi}\, \frac{\rho_q \omega}{2\cancel{\pi}} \int r^2\, d\tau$$

$$\therefore \boxed{\mu = \frac{\rho_q \omega}{2} \int r^2\, d\tau} \quad \text{(ii)}$$

Gyroscopic ratio: $\gamma = \dfrac{\mu}{L} = \dfrac{(ii)}{(i)}$

$$\therefore \gamma = \frac{\frac{\rho_q \cancel{\omega}}{2} \cancel{\int r^2 d\tau}}{\rho_m \cancel{\omega} \cancel{\int r^2 d\tau}} = \frac{\rho_q}{2\rho_m}$$

but $$\frac{\rho_q}{\rho_m} = \frac{Q/\cancel{V}}{M/\cancel{V}} = \frac{Q}{M}$$

$$\boxed{\therefore \gamma = \frac{Q}{2M}}$$

3. $$\gamma = \frac{\mu}{L} = \frac{\mu}{\omega I_{rot}}$$

$$\therefore \mu = \gamma \omega I_{rot}$$

$$= \frac{Q}{2M} \omega I_{rot}$$

(i) ring: $I_{rot} = MR^2$

$$\therefore \mu_{ring} = \frac{Q}{2\cancel{M}} \omega \cancel{M} R^2$$

$$\Rightarrow \boxed{\mu_{ring} = \frac{1}{2} Q \omega R^2}$$

(ii) disk: $I_{rot} = \frac{1}{2} MR^2$

$$\therefore \mu_{disk} = \frac{Q}{2M} \omega \frac{1}{2} MR^2$$

$$\Rightarrow \boxed{\mu_{disk} = \frac{1}{4} Q \omega R^2}$$

(iii) sphere : $I_{rot} = \frac{2}{5} M R^2$

$$\mu_{sphere} = \frac{Q}{\cancel{2}\cancel{M}} \omega \frac{\cancel{2}}{5} \cancel{M} R^2$$

$$\Rightarrow \boxed{\mu_{sphere} = \frac{1}{5} Q \omega R^2}$$

(iv) spherical shell : $I_{rot} = \frac{2}{3} M R^2$

$$\mu_{shell} = \frac{Q}{2M} \omega \frac{2}{3} M R^2$$

$$\Rightarrow \boxed{\mu_{shell} = \frac{1}{3} Q \omega R^2}$$

4. (a)

$$L = \omega I_{rot} = \frac{2}{5} m_e \omega r_e^2 = \frac{\hbar}{2}$$

$$\therefore \omega = \frac{5\hbar}{4 m_e r_e^2}$$

$$= \frac{5\hbar}{4m_e}\left(\frac{4\pi\varepsilon_0 m_e c^2}{e^2}\right)^2$$

$$= \frac{5\hbar m_e}{4}\left(\frac{4\pi\varepsilon_0 c^2}{e^2}\right)^2$$

$$\boxed{\therefore \omega = 1.828\times10^{25}\ s^{-1}}$$

(b)

A pt. on the surface of the sphere would move w/ speed

$$V = \omega r_e = 5.15\times10^{10}\ m/s$$

$$\boxed{\therefore V = 172c}\ !$$